Year 5 students' available and accessible knowledge of decimal-number numeration

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This paper reports on a study of the decimal-number numeration knowledge held by 112 Year 5 students who had completed formal instruction in tenths and hundredths. It explores the interaction between available and accessible place-value and regrouping knowledge. Available knowledge was elicited through diagnostic test items; accessible knowledge was elicited through an error analysis of the students' numeration procedures used in addition and subtraction algorithms. The results showed that performance varied markedly between classes (indicating an instructional effect), that regrouping was more difficult than place value, and that there was generally a direct relationship between available and accessible knowledge (although there were many instances of high-available and low-accessible knowledge).

According to Greeno and Riley (1987), the fundamental question of understanding a cognitive procedure is whether a person performs with some understanding or whether the performance is rote or mechanical. Correct responses may be based on inappropriate knowledge (e.g., claiming 4.52 is larger than 4.3 because it has more digits), and incorrect responses may be based on appropriate knowledge (e.g., knowing that 43 –17 is 4 less 43 –13 but counting back incorrectly to get 27 instead of 26). Therefore, performance alone is an insufficient indicator of the presence or absence of understanding. Moreover, because "one can never be certain whether impaired performance is due to faulty or absent use of well-articulated (memory) knowledge or 'efficient' use of inadequate knowledge" (Cavanaugh & Perlmutter, 1982, p. 15), it is difficult to distinguish between *unavailability* (a knowledge limitation) and *inaccessibility* (a performance limitation).

Bransford, Sherwood, Vye and Reiser (1986) argued that, whilst acquisition (availability) of relevant knowledge provides no guarantee of access, poor performance can often be attributed to access failure (not unavailability). Schoenfeld (1985) concurred: "The issue for students is often not how efficiently they will use the relevant resources potentially at their disposal. It is whether they will allow themselves access to those resources at all" (p. 13). Therefore, as Prawat (1989) claimed, teaching for accessibility is a much more complex process than teaching for availability (i.e., knowledge acquisition).

Decimal-number numeration knowledge. In her study of students' acquisition of, and access to, the cognitions required to function competently with decimal numbers, Baturo (1998) tested 173 Year 6 students from two schools (different socioeconomic backgrounds) with a pencil-and-paper diagnostic instrument to determine the students' available knowledge of the numeration processes, namely, number identification, place value, counting, regrouping, comparing, ordering, approximating and estimating for tenths and hundredths. As a result of analyses of the students' performances and of the cognitions embedded in decimal-number numeration processes, Baturo developed a numeration model (see Figure 1) to show these cognitions and how they may be connected.

The model depicts decimal-number numeration as having three levels of knowledge that are hierarchical in nature and therefore represent a sequence of cognitive complexity. *Level 1 knowledge* is the baseline knowledge associated with *position, base* and *order*, without which students cannot function with understanding in numeration tasks. Baseline knowledge is unary in nature comprising static memory-objects (Derry, 1996) from which all decimal-number numeration knowledge is derived. *Level 2 knowledge* is the "linking" knowledge associated with *unitisation* (Behr, Harel, Post & Lesh, 1994; Lamon, 1996) and *equivalence,* both of which are derived from the notion of base. It is binary in nature and therefore represents relational mappings (Halford, 1993). *Level 3 knowledge* is the structural knowledge that provides the superstructure for integrating all levels and is associated with *reunitisation*, *additive structure* and *multiplicative structure*. It incorporates ternary relations that are the basis of system mappings (Halford, 1993).

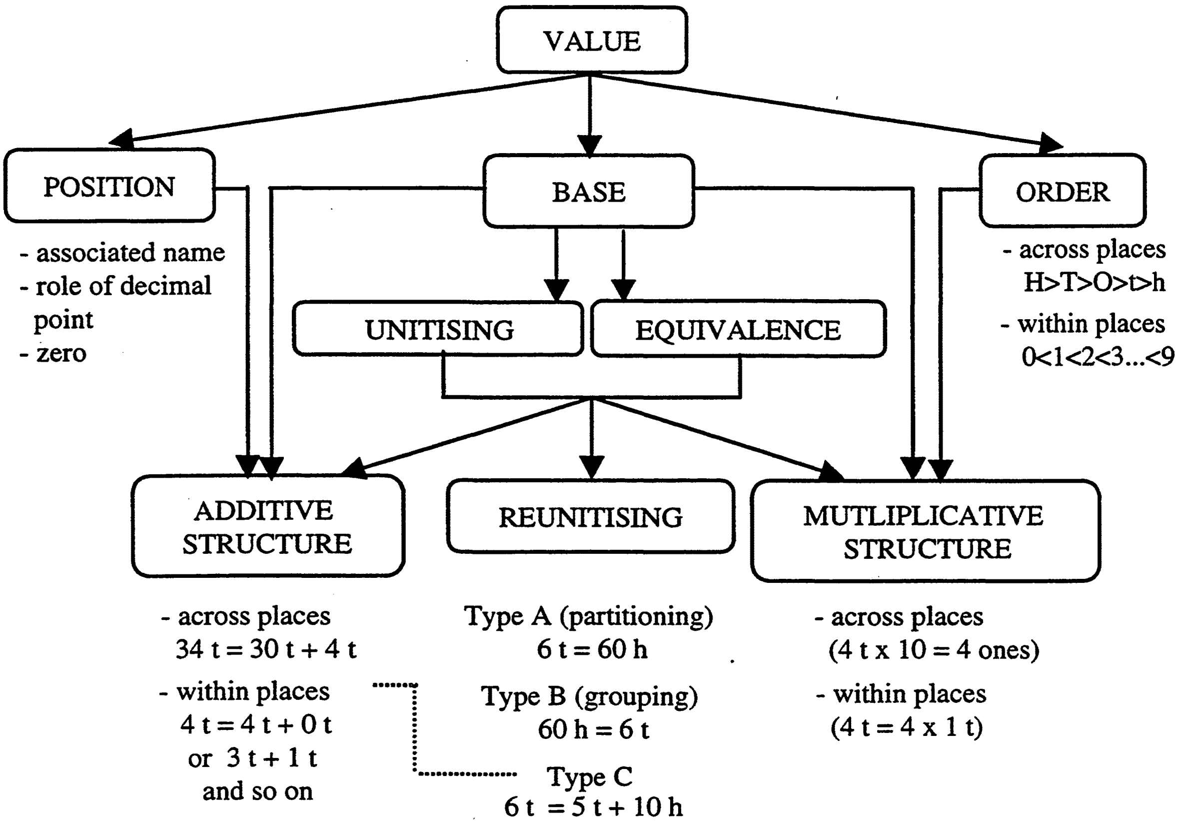

Figure 1. Cognitions and their connections embedded in the decimal number system (Baturo, 1997).

Within the model, *multiplicative structure* relates *position* and *base* into an exponential system (Behr, Harel, Post, & Lesh, 1994; Smith & Confrey, 1994) to give *value* and *order*. It is continuous and bi-directional and, for binary relationships, relates all adjacent places to the left through multiplication by 10 and to the right through division by 10. (For ternary relationships, it relates all adjacent-but-one places to the left through multiplication by 100 and to the right through division by 100.) It is the knowledge structure that underlies the concept of *place value*, the development of which is a major teaching focus in the primary school. This structure conflicts with the pattern of the place names which is syntactic in nature (about the ones), a conflict which is compounded if the decimal point is seen as having a position similar to ones (Baturo, 1997).

Also within the model, *unitisation, equivalence* and *reunitisation* underlie the decimal-number numeration processes of *renaming (*reunitisation Types A and B*)* and *regrouping (*reunitisation Type C*)*. These processes are particularly important in understanding decimal-number numeration beyond tenths (i.e., hundredths and thousandths). Decimal numbers can be named in standard form (e.g., 2.43 as 2 ones 4 tenths 3 hundredths), renamed (e.g., 2.43 as 2 ones 43 hundredths), and regrouped (e.g., 2.43 as 2 ones 3 tenths and 13 hundredths).

Available knowledge of decimal-number numeration can be directly probed, but accessible knowledge is mainly evident from situations where knowledge is applied. This paper reports on one study which compared Year 5 students' available knowledge of place value and regrouping for decimal numbers (limited to hundredths) as measured by diagnostic items with accessible knowledge as measured by error analysis of applications of this knowledge within addition and subtraction situations.

Method

Subjects: The study involved 112 Year 5 students from two schools each of which had three classes of Year 5 students. School A's clientele (Classes A, B, C) was drawn from a high socioeconomic background while School B's clientele (Classes D, E, F) was drawn from a low-to-middle socioeconomic background.

Instruments: The first instrument was a diagnostic test developed to cover all the numeration processes prescribed in the Queensland mathematics syllabus. These were categorised as *number identification* (identifying decimal numbers represented in pictorial, word and symbolic forms), *place value*, *regrouping*, *counting*, *comparing*, *ordering*, *approximating* and *estimating*. Because the numeration items were designed to probe available semantic knowledge, several different items were developed to assess each numeration component. The nonprototypic (Herhskowitz, 1989) items designed to create "expectation violations" (Schank, 1986) yielded insight into the robustness of the students' available knowledge of place value and regrouping (see Figure 2).

PLACE VALUE		REGROUPING	
Write the number that has:	______	Write the numbers.	
2 tenths, 5 hundredths, 4 ones	______	86 tenths 3 hundredths	______
19 ones, 2 hundredths	______	7 tenths 14 hundredths	______
7 ones, 4 tenths	______		
3 hundredths, 6 tenths	______		
8 tenths	______		
5 hundredths	______		

Figure 2. Test items to probe decimal-number place value and regrouping.

The second instrument comprised two computation tasks. The focus of these tasks was to determine the accessibility of the students' place value and regrouping knowledge, the availability of which was determined from the diagnostic tasks (see Figure 2). The addition task (23.5 + 3.67) was designed to probe the application of decimal-number place value knowledge. For example, it was anticipated that students with low available place value knowledge would merely align the digits from left to right or from right to left without any regard to the values of the digits. This task was thus seen as a means of determining whether students with high available place value knowledge would transfer this knowledge to a different task (and therefore indicating a metacognitive awareness as well as a procedural awareness of place value). Whilst there was also a regrouping component, this was not considered to be as cognitively demanding as the regroupings required in the subtraction task.

The subtraction task (8.2 - 1.76) was designed to probe accessible decimal-number regrouping knowledge with respect to knowing **when** to regroup (e.g., when having to subtract the 6 hundredths) and **how** to regroup prototypic instances (e.g., when having to subtract the 7 tenths). Whilst there was also a place value component to this task, it was anticipated that students with low available place value knowledge would either write the "larger number" (i.e., 1.76) on top (and therefore have no chance of a correct solution) or align the places from left to right, a syntactic procedure that would produce the correct alignment without any regard to knowing that like places should be aligned.

Procedure: The diagnostic test was administered, at the same time, to all students within each school with the tasks being read to students to overcome possible reading problems. The computation tasks were administered to the students following the test. Students were required to record their working for the computation tasks. Most students completed the test and the computation tasks in about 20 minutes.

Analysis: Individual and class performances (available knowledge from the diagnostic test) were collated with respect to place value and regrouping and then individual performance was categorised as *low, medium, or high.* The classification criteria for the 6 place value items and the 2 regrouping items were: *low* (place value – ≤2; regrouping – 0); *medium* (place value – 3 or 4; regrouping – 1); *high* (place value – ≥5; regrouping – 2). For the computation items, performance was analysed in terms of flow charts that showed all options with respect to place value and regrouping in the solution procedures. Comparisons were then made between each student's available and accessible place value and regrouping knowledge.

Results

Table 1 provides the class means for the place value and regrouping components of the test and the computation tasks. As shown in this table, there were major class and school differences within and between all components. (Independent Sample T-tests revealed that many differences were significant at $p<0.001$.) In particular, performances were more impoverished for subtraction than for addition.

Table 1
Class Means (%) for Place Value, Regrouping and Computation.

	School A			School B		
Items	Class A	Class B	Class C	Class D	Class E	Class F
Place value	58.9	44.9	78.9	31.0	28.9	54.7
Regrouping	40.8	37.5	43.6	28.1	42.6	40.9
23.5+3.67	73.7	68.8	62.5	62.5	35.3	81.8
8.2–1.76	15.8	37.5	31.3	04.2	00.0	36.4

Available knowledge: From the test results, students were classified as high, medium and low with respect to available place value and regrouping knowledge. Table 2 shows the number and percentage of students within each categorisation. All but one high regrouping student were also high place value students. Approximately one-third (23) of the high place value students revealed low regrouping knowledge.

Table 2 reveals a markedly different category distribution for available place value knowledge than for available regrouping knowledge. The number of high available regrouping students is significantly less than the number of high available place value students. This indicates the inherent complexity in regrouping – having high available place value knowledge appears to be an essential but not sufficient prerequisite for high regrouping knowledge.

Table 2
Number and Percent of Students in Each Available Knowledge Category with Respect to Place Value and Regrouping Knowledge.

	Available knowledge categories		
Knowledge	Low	Medium	High
Place value	37 (33.0%)	17 (15.2%)	58 (51.8%)
Regrouping	59 (52.7%)	42 (37.5%)	11 (09.8%)

Accessible knowledge: Details of the students' performances on the two computation tasks are shown on the flow charts in Figures 3 and 4.

Figure 3 reveals that, as expected, the most critical point in the solution of this addition task which involved decimal numbers with different whole-number "lengths" was in the alignment of the numbers. Only one third of the students knew to align and add like digits. Of these 77 students, 49 (representing 84.5% of their category) had exhibited high available place value knowledge, 12 (representing 70.6% of their category) had exhibited medium available place value knowledge, and 15 (representing 40.5% of their category) had exhibited low available place value knowledge. These results indicate that having high available place value knowledge is not a sufficient indicator of having accessible knowledge

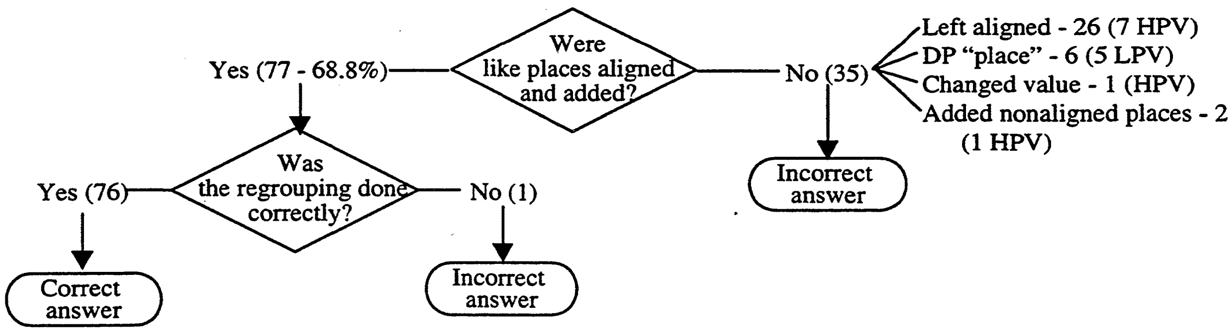

Figure 3: Students' performances on the addition task (23.5 + 3.67).
(HPV = high available place value knowledge; LPV = low available place value knowledge.)

Figure 4 reveals that 12 students who had exhibited high available place value knowledge (HPV) were unable to access this knowledge when preparing the decimal numbers for subtraction. The HPV student who had aligned the subtraction numbers incorrectly had also aligned the addition numbers incorrectly. Figure 4 also reveals the complexity of regrouping (in knowing when and how to regroup) with only 21 of the 73 students (28.8%) who had prepared the numbers correctly being able to complete the regrouping component correctly. Of these 21 students, 4 (representing 36.4% of their category) had exhibited high available place value knowledge, 13 (representing 31.0% of their category) had exhibited medium regrouping knowledge, and 4 (representing 06.8%) of their category) had exhibited low available regrouping knowledge.

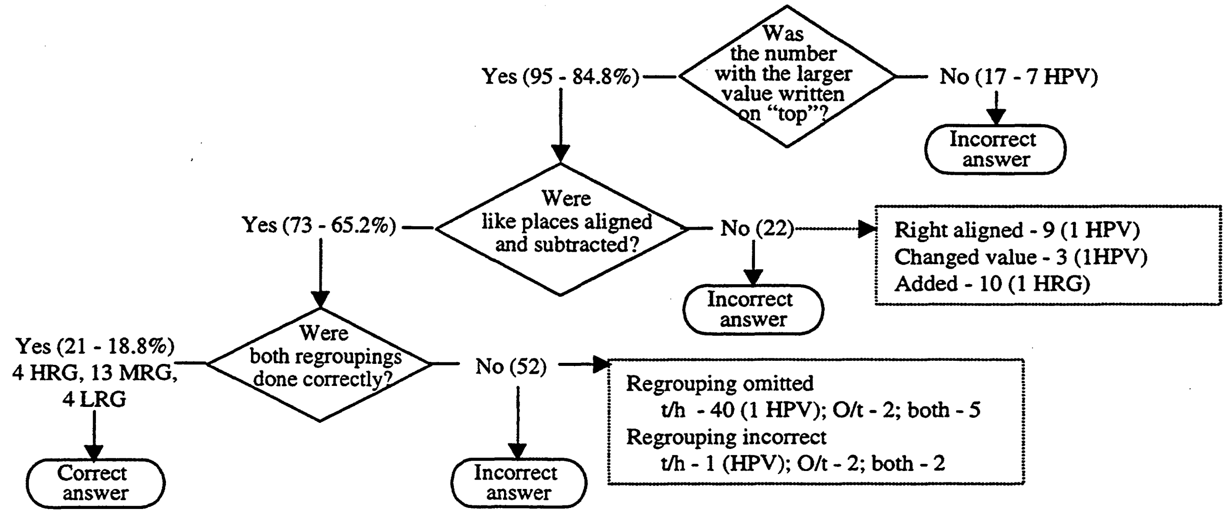

Figure 4: Students' performances on the subtraction task (8.2 – 1.76).
(HPV = high available place value knowledge; HRG/MRG/LRG = high/medium/low available regrouping knowledge.)

Comparing available and accessible knowledge: Table 3 provides a cross-tabulation of students' available and accessible place value and regrouping knowledge with respect to the performance categories. It presents accessible regrouping data as Regrouping P (referring to the prototypic, that is, standard/"simple" regrouping) and Regrouping NP (referring to the nonprototypic, that is, involving an "understood" zero regrouping). Table 3 also contains reductions in the response totals for regrouping because incorrect setting up of the subtraction task resulted in removing the need for regrouping. It also includes the results of the 29 students (20 low, 8 medium, 1 high) who had not prepared the subtraction correctly but had encountered an instance where regrouping (mainly prototypic) was required. Of these 29 students, 15 (11 low, 4 medium) indicated that they knew when and how to regroup.

Table 3 shows a strong direct relationship between available and accessible place value knowledge, a modest relationship for accessible nonprototypic regrouping, but no relationship for accessible prototypic regrouping. For both regroupings, the relationship was not supported for high available knowledge. This may have been a consequence of the distribution difference between available place value and regrouping knowledge (see Table 2).

Table 3
Percent and Number of Students' Accessible Place Value and Regrouping Knowledge With Respect to the Available Knowledge Categories.

	Available knowledge categories		
Accessible knowledge	Low	Medium	High
Place value	45.9% (17/37)	70.6% (12/17)	87.9% (51/58)
Regrouping			
– RGP	62.7% (37/59)	78.6% (33/42)	63.6% (7/11)
–RGNP	10.2% (6/59)	42.9% (18/42)	45.5% (5/11)

Note. RGP = prototypic regrouping; RGNP = nonprototypic regrouping.

Discussion and conclusion

Students' performance on place-value, regrouping and computation items varied markedly between classes and schools (see Table 1), but overall, performance was better on place-value items than regrouping (see Table 2). It was also evident that regrouping was much more difficult for subtraction than for addition (see Figures 3 and 4). The significant differences within and between classes and schools could be attributed to instructional experiences, particularly to certain surface forms that, for some classes, appeared to have been over-practised (e.g., inserting a "rightmost zero"). The test revealed that the prerequisite whole-number and fraction concepts and processes (e.g., Resnick et al., 1989) were impoverished in some classes. Effective instruction needs to focus on establishing the prerequisite knowledge and by ensuring that prior learning is connected to new learning. Special attention needs to be given to place value, particularly for decimal numbers of unequal length, and for regrouping, particularly for subtraction.

The results for regrouping warrant an investigation into the mathematical structure of both types of regrouping and into how students acquire and access both types, particularly when internal and rightmost zeros are involved. Figure 1 shows that regrouping is a Type C reunitising construct that has been classified as Level 3 knowledge (structural knowledge). Level 3 knowledge encompasses ternary relationships (Halford, 1993) which are more cognitively complex than the binary or unary relationships of Levels 2 and 1 respectively. Type C reunitisation is more difficult than Types A and B because the decomposition within the given unit links it to additive structure (another Level 3 knowledge type). The complexity of regrouping and the current curriculum trend promoting mental and calculator computation raise pedagogical issues. For example, should time-consuming pencil-and-paper computation be eliminated from modern curricula?

This paper takes the stance that there is a place for all types of computation. Calculators and/or mental computation should be employed when speedy and accurate calculations are required, and pencil-and-paper computation should be employed as a vehicle for promoting understanding of mathematical principles rather than procedural proficiency. For instance, underlying pencil-and-paper computation are mathematical principles (e.g., set inclusion which develops the understanding that like things only can be added and subtracted; the commutative and distributive laws) that apply across whole-number, fraction and algebra domains. The structural knowledge embedded in computational knowledge suggests that a study of computation (not merely computational procedures for "getting an answer") should be undertaken at the upper primary and lower secondary levels.

The analysis of the students' computations indicated the existence of rules that appeared not to have been identified by the literature. For example, several students appeared to align numbers from the left, irrespective of place value. As well, many students seemed to employ an "equalising" procedure to ensure that the given numbers had the same "length". Whilst most students inserted a rightmost zero (which did not change the value of the given number), some children inserted a zero internally (i.e., immediately after the decimal point) so that the value was changed. These latter students may employ a rule "zero has no value" no matter where it is placed.

Whilst the data indicated that there was generally a direct relationship between available and accessible knowledge (see Table 3), it also indicated what is evident to any teacher – that correct individual performance is possible without understanding and that availability of knowledge does not mean it will be used (accessed). Students who failed to access their available knowledge may have lacked awareness (a key metastrategy in Prawat's, 1989, theory of accessibility) of the role of place value and regrouping in computation.

The low-available knowledge students who "accessed" unavailable appeared to be following procedures routinised from overpractised whole-number procedures (Hiebert & Wearne, 1985), thus bypassing the need to access available semantic knowledge by replacing it with now available syntactic knowledge. It seems doubtful that these students demonstrated a semantic understanding of the role of place value in the addition operation and it seems more likely that they had applied the rule "line up the decimals". As well, there were medium- and high-available place-value students who did not correctly align the decimal numbers in either computation. Similarly, there were correct low-available and incorrect high-available regrouping students. In several cases, errors resulted from failure to activate a procedure rather than from incorrect execution of the procedure.

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